



دفتر : كالculus 2 calculus 2

ورد طعمه

للطالب

اللجنة الأكاديمية لقسم الهندسة الصناعية

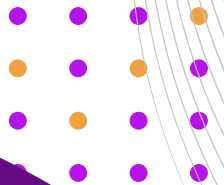
2025



Turbo IEG.Com



Turbo Team Youtube



Integration by parts:-

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\frac{d}{dx} (f(x) \cdot g(x)) = f'(x) g(x) + f(x) g'(x)$$

$$\int \frac{d}{dx} (f(x) g(x)) dx = \int f'(x) g(x) dx + \int f(x) g'(x) dx$$

$$f(x) = u \quad g(x) = v$$

$$f(x) g(x) = \int f'(x) g(x) dx = \int f(x) g'(x) dx$$

$$u \cdot v - \int v \cdot du = \int u \cdot dv$$

$$\begin{array}{ccc} u & & dv \\ \downarrow \text{نشتقها} & \searrow & \downarrow \text{نضربها} \\ du & \xrightarrow{-\int} & v \end{array} \quad \leftarrow \int u \cdot dv$$

$$= u \cdot v - \int v du$$

Example :- $\int \ln x dx$

$$\begin{array}{ccc} u = \ln x & & dv = dx \\ \downarrow & \searrow & \downarrow \\ du = \frac{1}{x} dx & & v = x \end{array}$$

$$= \ln x \cdot x - \int \frac{1}{x} dx \cdot x$$

$$= x \ln x - x + c$$

Integration by part

للتسهيل

* $\int \frac{\text{أقتران}}{dv} \frac{\ln(u)}{u}$ * $\int \frac{\text{كثير حدود}}{u} \frac{\cos(\text{خطي})}{\sin(\text{خطي})} dv$

* $\int \frac{\text{أقتران}}{dv} \frac{\sin^{-1}(u)}{\tan^{-1}(u)}$ * $\int \frac{\text{خطي}}{dv} \frac{\sin(\text{خطي})}{u}$

* $\int \frac{\text{كثير حدود}}{u} \frac{e^{\text{خطي}}}{dv}$ * $\int \frac{\text{poly}}{u} \frac{(\text{خطي})^n}{dv}$

Example: $\int_0^1 \ln(x^2+1) dx$

$u = \ln(x^2+1) \quad dv = dx$
 $du = \frac{2x}{x^2+1} \quad v = x$

$$= x \ln(x^2+1) - \int \frac{2x}{x^2+1} dx$$

$$= x \ln(x^2+1) - 2 \int \frac{(x^2+1)-1}{x^2+1} dx$$

$$= x \ln(x^2+1) - 2 \int 1 - \frac{1}{x^2+1} dx$$

$$= x \ln(x^2+1) - 2 \left(x - \tan^{-1} x \right) \Big|_0^1$$

$$= \left(\ln 2 - 2(1 - \tan^{-1}(1)) \right) - \left(0 \ln 1 - 2(0 - \tan^{-1}(0)) \right)$$

$$= \ln 2 - 2 + \frac{2\pi}{4} - 0$$

$$= \ln 2 - 2 + \frac{\pi}{2}$$

Example :- $\int \cos x \cdot \ln(\cos x) dx$

$$u = \ln(\cos x) \quad dv = \cos x$$

$$du = \frac{-\sin x dx}{\cos x} \quad v = \sin x$$

$$= \sin x \times \ln(\cos x) + \int \sin x \tan x dx$$

$$= \sin x \times \ln(\cos x) + \int \frac{1}{\cos x} - \cos x dx$$

$$= \sin x \times \ln(\cos x) + \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx - \sin x$$

$$= \sin x \times \ln(\cos x) - \sin x + \ln |\sec x + \tan x| + c$$

Example :- $\int_0^{1/3} \tan^{-1}(3x) dx$

$$u = \tan^{-1}(3x) \quad dv = dx$$

$$du = \frac{3 dx}{1+9x^2} \quad v = x$$

$$= x \times \tan^{-1}(3x) - \int \frac{3x}{1+9x^2} dx$$

$$= x \times \tan^{-1}(3x) - \frac{1}{6} \int \frac{18x}{1+9x^2} dx$$

$$= x \times \tan^{-1}(3x) - \frac{1}{6} \ln |1+9x^2| \Big|_0^{1/3}$$

$$= \left(\frac{1}{3} \tan^{-1}(1) - \frac{1}{6} \ln |2| \right) - 0$$

$$= \frac{1}{3} \frac{\pi}{4} - \frac{1}{6} \ln 2$$

$$\frac{\pi}{12} - \frac{1}{6} \ln 2$$

Example:- $\int x^2 e^{-x} dx$

$$\begin{array}{ll} u = x^2 & dv = e^{-x} \\ du = 2x dx & v = -e^{-x} \end{array}$$

-f

$$= -x^2 e^{-x} - \int -2x e^{-x} dx$$

$$= -x^2 e^{-x} + \int 2x e^{-x} dx$$

$$= -x^2 e^{-x} - 2x e^{-x} - \int -2e^{-x} dx$$

$$\begin{array}{ll} u = 2x & dv = e^{-x} \\ du = 2 dx & v = -e^{-x} \end{array}$$

-f

$$= -x^2 e^{-x} - 2x e^{-x} + 2e^{-x} + C$$

لمربوطة الجدول

$$-x^2 e^{-x} - 2x e^{-x} + 2e^{-x} + C$$

u	dv
x^2	e^{-x}
$2x$	$-e^{-x}$
2	e^{-x}
0	$-e^{-x}$

-f

Example: $\int x^2 \sin(2x) dx$

	u	dv
	x^2	$\sin(2x)$
	$2x$	$-\frac{1}{2} \cos(2x)$
	2	$-\frac{1}{4} \sin(2x)$
	0	$+\frac{1}{8} \cos(2x)$

$$= -\frac{x^2 \cos(2x)}{2} + \frac{2x \sin(2x)}{4} + \frac{2 \cos(2x)}{8} + C$$

$$= -\frac{x^2 \cos(2x)}{2} + \frac{x \sin(2x)}{2} + \frac{\cos(2x)}{4} + C$$

Example: $\int e^{2x} \sin x dx$

	u	dv
	e^{2x}	$\sin x$
	$2e^{2x}$	$-\cos x$
	$4e^{2x}$	$+\sin x$

$$\int e^{2x} \sin x dx = -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x dx$$

$$5 \int e^{2x} \sin x dx = -e^{2x} \cos x + 2e^{2x} \sin x$$

$$\int e^{2x} \sin x dx = -\frac{1}{5} e^{2x} \cos x + \frac{2}{5} e^{2x} \sin x + C$$

11

$u = e^{2x}$	$dv = \sin x$
$du = 2e^{2x} dx$	$v = -\cos x$

Example 8-

$$11) \int x^3 \sin x^2 dx$$

$$\text{let } y = x^2$$

$$dy = 2x dx$$

$$dx = \frac{dy}{2x}$$

$$\int x^3 \sin y \frac{dy}{2x}$$

$$\frac{1}{2} \int x^2 \sin y dy$$

$$\frac{1}{2} \int y \sin y dy$$

u	dV
y	$\sin y$
1	$-\cos y$
0	$-\sin y$

+ $\int$

$$= \frac{1}{2} (-y \cos y + \sin y) + C$$

$$= \frac{1}{2} (-x^2 \cos x^2 + \sin x^2) + C$$

$$12) \int \cos(\ln x) dx$$

$$e^y = x$$

$$\text{let } y = \ln x$$

$$dy = \frac{1}{x} dx$$

$$\int \cos y x dy$$

$$\int \cos y e^y dy$$

$$dx = x dy$$

$$\int \cos y e^y dy = e^y \sin y + e^y \cos y + \int e^y \cos y dy$$

u	dV
e^y	$\cos y$
e^y	$\sin y$
e^y	$-\cos y$

+ $\int$

$$2 \int \cos y e^y dy = e^y (\sin y + \cos y)$$

$$\int \cos y e^y dy = \frac{e^y}{2} (\sin y + \cos y) + C$$

$$= \frac{e^{\ln x}}{2} (\sin(\ln x) + \cos(\ln x)) + C$$

$$= \frac{x}{2} (\sin(\ln x) + \cos(\ln x)) + C$$

Q3] suppose that $f(1)=2$, $f(4)=7$, $f'(1)=5$
 $f'(4)=3$ and f'' is cont. Find the value of $\int_1^4 x f''(x) dx$

$$\int_1^4 x f''(x) dx$$

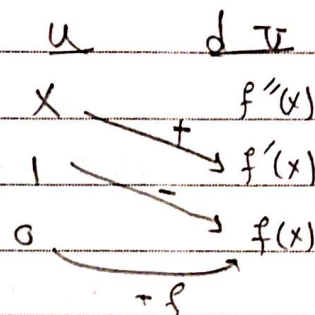
$$= \left[x f'(x) - f(x) \right]_1^4$$

$$= (4 f'(4) - f(4)) - (1 f'(1) - f(1))$$

$$= (12 - 7) - (5 - 2)$$

$$= 5 - 3$$

$$= 2$$



Trigonometric Integrals

$$* \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$* \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Example :- prove the reduction formula

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$$

$$\begin{aligned} u &= \cos^{n-1} x & dV &= \cos x \, dx \\ du &= (n-1) \cos^{n-2} x (-\sin x) & \sqrt{V} &= \sin x \end{aligned}$$

$$= \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x (\sin x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x - \cos^n x \, dx$$

$$\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$1 + n - 1 \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Example 2:

$$(1) \int \sin^4 x \, dx$$

$$= -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx$$

يمكن استخدامه
 $(\sin^2 x = \frac{1}{2}(1 - \cos 2x))$

$$= -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx$$

$$= -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left(-\frac{1}{2} \sin x \cos x + \frac{1}{2} \int \sin^2 x \, dx \right)$$

$$= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C$$

$$(2) \int \cos^3(3x) \, dx = \frac{1}{3} \int \cos^3(y) \, dy$$

$$\text{let } y = 3x$$

$$dy = 3 \, dx$$

$$= \frac{1}{3} \left(\frac{1}{3} \cos^2 y \sin y + \frac{2}{3} \int \cos y \, dy \right) \quad dx = \frac{2}{dy}$$

$$= \frac{1}{9} \cos^2 y \sin y + \frac{2}{9} \sin y + C$$

$$= \frac{1}{9} \cos^2(3x) \sin(3x) + \frac{2}{9} \sin(3x) + C$$

strategy for Evaluating $\int \sin^m x \cos^n x dx$

a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factor in terms of sine:

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x (\cos^2 x)^k \cos x dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x dx \end{aligned}$$

The substitute $u = \sin x$

b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factor in terms of cosine:

$$\begin{aligned} \int \sin^{2k+1} x \cos^n x dx &= \int (\sin^2 x)^k \cos^n x \sin x dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x dx \end{aligned}$$

The substitute $u = \cos x$ [Note: - that if the powers of both sine and cosine are odd either (a) or (b) can be used]

c) If the power of both sine and cosine are even use the half angle identities

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x) \quad \cos^2 x = \frac{1}{2} (1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Example 8-

$$\text{Q1} \int \sin^3 x \cos^2 x \, dx$$

$$= \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

$$= \int (1 - u^2) u^2 \sin x \frac{du}{-\sin x}$$

$$= \int u^4 - u^2 \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{1}{5} (\cos x)^5 - \frac{1}{3} (\cos x)^3 + C$$

$$u = \cos x$$

$$du = -\sin x \, dx$$

$$dx = \frac{du}{-\sin x}$$

$$\text{Q2} \int \sin^6(5x) \cos^3(5x) \, dx$$

$$= \frac{1}{5} \int \sin^6 y \cos^3 y \, dy$$

$$= \frac{1}{5} \int \cos^2 y \sin^6 y \cos y \, dy$$

$$= \frac{1}{5} \int (1 - \sin^2 y) \sin^6 y \cos y \, dy$$

$$= \frac{1}{5} \int (1 - u^2) u^6 \, du$$

$$= \frac{1}{5} \int u^6 - u^8 \, du$$

$$= \frac{1}{5} \left(\frac{1}{7} (\sin 5x)^7 - \frac{1}{9} (\sin 5x)^9 \right)$$

$$= \frac{1}{35} \sin^7 5x - \frac{1}{45} \sin^9 5x + C$$

$$y = 5x$$

$$\frac{dy}{5} = dx$$

$$u = \sin y$$

$$du = \cos y \, dy$$

$$dy = \frac{du}{\cos y}$$

$$Q3) \int \sin^2 x \cos^4 x \, dx = \int \frac{1}{2} (1 - \cos 2x) \left(\frac{1}{2} (1 + \cos 2x) \right)^2 dx$$

$$= \int \frac{1}{2} (1 - \cos 2x) \left(\frac{1}{4} (1 + 2 \cos 2x + \cos^2 2x) \right) dx$$

$$= \frac{1}{8} \int 1 + 2 \cos(2x) + \cos^2(2x) - \cos(2x) - 2 \cos^2(2x) - \cos^3(2x) dx$$

$$= \frac{1}{8} \int 1 + \cos(2x) - \cos^2(2x) - \cos^3(2x) \, dx$$

$$= \frac{1}{8} \left(x + \frac{\sin 2x}{2} - \int \frac{1}{2} (1 + \cos 4x) \, dx - \frac{1}{2} \int \cos^3 y \, dy \right)$$

$$= \frac{1}{8} \left(x + \frac{\sin 2x}{2} - \frac{1}{2} x - \frac{1}{8} \sin 4x - \frac{1}{6} \cos^2(2x) \sin(2x) - \frac{2}{6} \sin(2x) \right) + C$$

* To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$

(b) $\int \sin mx \sin nx \, dx$ OR

(c) $\int \cos mx \cos nx \, dx$

use the corresponding identity :-

$$a) \sin A \cos B = \frac{1}{2} (\sin(A-B) + \sin(A+B))$$

$$b) \sin A \sin B = \frac{1}{2} (\cos(A-B) - \cos(A+B))$$

$$c) \cos A \cos B = \frac{1}{2} (\cos(A-B) + \cos(A+B))$$

Example e-

$$11) \int \cos(4x) \sin(3x) dx$$

$$= \frac{1}{2} \int \sin(3x - 4x) + \sin(3x + 4x) dx$$

$$= \frac{1}{2} \int -\sin(x) + \sin(7x) dx$$

$$= \frac{1}{2} \int \sin(7x) - \sin(x) dx$$

$$= \frac{1}{2} \left(\cos(x) - \frac{1}{7} \cos(7x) \right) + C$$

$$12) \int \cos(2x) \cos(4x) dx$$

$$= \frac{1}{2} \int \cos(2x - 4x) + \cos(2x + 4x) dx$$

$$= \frac{1}{2} \int \cos(2x) + \cos(6x) dx$$

$$= \frac{1}{2} \left(\frac{1}{2} \sin(2x) + \frac{1}{6} \sin(6x) \right) + C$$

$$* \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

$$* \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

$$\begin{aligned} * \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = - \int \frac{-\sin x}{\cos x} \, dx \\ &= - \ln |\cos x| \\ &= + \ln |\sec x| \end{aligned}$$

$$\begin{aligned} * \int \sec x \, dx &= \int \sec x (\sec x + \tan x) \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\ &= \ln |\sec x + \tan x| + C \end{aligned}$$

Example 8 -

$$\text{ii) } \int \tan^2(2x) \, dx$$

$$y = 2x$$

$$\frac{dy}{2} = dx$$

$$= \frac{1}{2} \int \tan^2 y \, dy$$

$$= \frac{1}{2} \left(\frac{\tan y}{1} - \int \tan y \, dy \right)$$

$$= \frac{1}{2} \tan y - \frac{1}{2} \ln |\sec y| + C$$

$$= \frac{1}{2} \tan(2x) - \frac{1}{2} \ln |\sec 2x| + C$$

$$12) \int \sec^4 x \, dx$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3} \int \sec^2 x \, dx$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3} \tan x + C$$

✱ strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

1a) If the power of secant is even ($n = 2k, k \geq 2$)
save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$
to express the remaining factors in terms of $\tan x$ -

$$\int \tan^m x \sec^n x \, dx = \int (\sec^2 x)^{k-1} \tan^m x \sec^2 x \, dx$$

$$= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$

1b) If the power of tangent is odd ($m = 2k+1$), save
a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to
express the remaining factors in terms of $\sec x$ -

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

The substitute $u = \sec x$

Example :-

$$[1] \int \tan^5 x \sec x \, dx$$

$$= \int \tan^4 x \sec x \tan x \, dx$$

$$= \int (\tan^2 x)^2 \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^2 \sec x \tan x \, dx$$

$$= \int (u^2 - 1)^2 \sec x \tan x \, dx$$

$$u = \sec x$$

$$du = \sec x \tan x \, dx$$

$$\frac{du}{\sec x \tan x} = dx$$

$$= \int (u^4 - 2u^2 + 1) \, du$$

$$= \frac{u^5}{5} - \frac{2u^3}{3} + u + C$$

$$= \frac{1}{5} (\sec x)^5 - \frac{2}{3} (\sec x)^3 + \sec x + C$$

$$[2] \int \tan^3 x \sec^4 x \, dx$$

$$= \int \tan^3 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^3 x (1 + \tan^2 x) \sec^2 x \, dx$$

$$u = \tan x$$

$$= \int u^3 (1 + u^2) \, du$$

$$\frac{du}{\sec^2 x} = dx$$

$$= \int u^3 + u^5 \, du$$

$$= \frac{1}{4} (\tan x)^4 + \frac{1}{6} (\tan x)^6 + C$$

$$13) \int \tan^4 x \sec x \, dx$$

$$\int (\tan^2 x)^2 \sec x \, dx$$

$$\int (\sec^2 x - 1)^2 \sec x \, dx$$

$$\int \sec^5 x - 2\sec^3 x + \sec x \, dx$$

العمل على القوانين
...

$$* \int \cot x \, dx = \ln |\sin x| + C$$

$$* \int \csc x \, dx = -\ln |\csc x + \cot x| + C$$

$$= \ln |\csc x - \cot x| + C$$

$$* \int \cot^2 x = \csc^2 x - 1$$

* Table to trigonometric Substitutions 8-

Expression	substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Example 8-

$$\text{11) } \int \frac{\sqrt{9 - x^2}}{x^2} dx$$

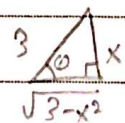
$$= \int \frac{\sqrt{3^2 - x^2}}{x^2} dx$$

let $x = 3 \sin \theta$
 $dx = 3 \cos \theta d\theta$

$$= \int \frac{\sqrt{9 - (3 \sin \theta)^2}}{(3 \sin \theta)^2} \cdot 3 \cos \theta d\theta$$

$\frac{\text{مقابل}}{\text{وتر}} = \frac{x}{3} = \sin \theta$

$$= \int \frac{\sqrt{9 - 9 \sin^2 \theta}}{9 \sin^2 \theta} \cdot 3 \cos \theta d\theta$$



$\sin^{-1}\left(\frac{x}{3}\right) = \theta$

$$= \int \sqrt{1 - \sin^2 \theta} \cdot 3 \cos \theta d\theta$$

$$= \int \frac{\sqrt{1 - \sin^2 \theta}}{\sin^2 \theta} \cdot \cos \theta d\theta = \int \frac{|\cos \theta| \cos \theta}{\sin^2 \theta} d\theta$$

$$= \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta = \int (\csc^2 \theta - 1) d\theta \rightarrow$$

$$= -\cot \theta - \theta + C$$

$$= -\cot \left(\sin^{-1} \left(\frac{x}{3} \right) \right) - \sin^{-1} \left(\frac{x}{3} \right) + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1} \left(\frac{x}{3} \right) + C$$

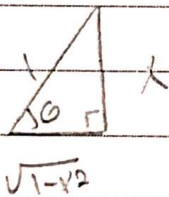
Example:-

$$\text{Find } \int \frac{1}{(1-x^2)^{3/2}} dx = \int \frac{1}{(\sqrt{1-x^2})^3} dx$$

$$\text{let } x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$= \int \frac{1}{\sqrt{(1-\sin^2 \theta)}^3} \cos \theta d\theta$$



$$= \int \frac{\cos \theta}{(\sqrt{\cos^2 \theta})^3} d\theta$$

$$\theta = \sin^{-1}(x)$$

$$= \int \frac{\cos \theta}{|\cos \theta|^3} d\theta$$

$$= \int \frac{1}{\cos^2 \theta} d\theta$$

$$= \int \sec^2 \theta d\theta$$

$$= \tan \theta + C$$

$$= \tan(\sin^{-1}(x)) + C$$

$$= \frac{x}{\sqrt{1-x^2}} + C$$

Trigonometric Substitution

lecture 8

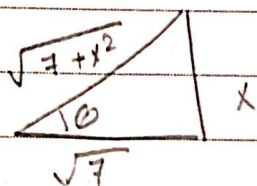
$$\text{B3) } \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{7+x^2}} dx = \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{(\sqrt{7})^2 + x^2}} dx$$

$$\text{Let } \begin{cases} x = \sqrt{7} \tan \theta \\ dx = \sqrt{7} \sec^2 \theta d\theta \end{cases}$$

$$\int \frac{(\sqrt{7} \tan \theta)^2 \times \sqrt{7} \sec^2 \theta d\theta}{\sqrt{7 + (\sqrt{7} \tan \theta)^2}}$$

$$\frac{x}{\sqrt{7}} = \tan \theta$$

$$\int \frac{7 \tan^2 \theta \times \sqrt{7} \sec^2 \theta d\theta}{\sqrt{7 + 7 \tan^2 \theta}}$$



$$\int \frac{7 \sqrt{7} \tan^2 \theta \sec^2 \theta d\theta}{\sqrt{7} \sqrt{1 + \tan^2 \theta}}$$

$$\theta = \tan^{-1}\left(\frac{x}{\sqrt{7}}\right)$$

$$\int \frac{7 \tan^2 \theta \sec^2 \theta d\theta}{\sqrt{\sec^2 \theta}}$$

$$\int 7 \tan^2 \theta \sec \theta d\theta$$

$$\int 7 (\sec^2 \theta - 1) \sec \theta d\theta$$

$$\int 7 (\sec^3 \theta - \sec \theta) d\theta$$

$$= 7 \left(\frac{\sec \theta \tan \theta}{2} + \frac{1}{2} \int \sec \theta d\theta - \int \sec \theta d\theta \right)$$

$$= 7 \left(\frac{\sec \theta \tan \theta}{2} - \frac{1}{2} \ln |\tan \theta + \sec \theta| \right)$$

$$= \frac{7}{2} \left(\frac{\sqrt{7+x^2}}{\sqrt{7}} \times \frac{x}{\sqrt{7}} - \ln \left| \frac{x}{\sqrt{7}} + \frac{\sqrt{7+x^2}}{\sqrt{7}} \right| \right)$$

$$= \frac{7}{2} \left(\frac{\sqrt{7+2}}{\sqrt{7}} \times \frac{\sqrt{2}}{\sqrt{7}} - \ln \left| \frac{\sqrt{2}}{\sqrt{7}} + \frac{\sqrt{7+2}}{\sqrt{7}} \right| - (0) \right)$$

$$= \frac{7}{2} \left(\frac{3\sqrt{2}}{7} - \ln \left(\frac{\sqrt{2}+3}{\sqrt{7}} \right) \right)$$

Example 8-

$$\text{Find } \int \frac{\sqrt{x^2 - 25}}{x} dx$$

$$= \int \frac{\sqrt{(5 \sec \theta)^2 - 25} \times 5 \sec \theta \tan \theta d\theta}{5 \sec \theta}$$

let $x = 5 \sec \theta$

$$= \int \sqrt{25 \sec^2 \theta - 25} \times \tan \theta d\theta$$

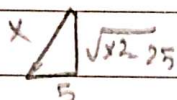
$dx = 5 \sec \theta \tan \theta d\theta$

$$= \int \sqrt{25} \sqrt{\sec^2 \theta - 1} \times \tan \theta d\theta$$

$\frac{x}{5} = \sec \theta$

$$= 5 \int |\tan \theta| \times \tan \theta d\theta$$

$\theta = \sec^{-1}\left(\frac{x}{5}\right)$



$$= 5 \int \sec^2 \theta - 1 d\theta$$

$$= 5 (\tan \theta - \theta) + C$$

$$= 5 \left(\tan \left(\sec^{-1} \left(\frac{x}{5} \right) \right) - \sec^{-1} \left(\frac{x}{5} \right) \right) + C$$

$$= 5 \left(\frac{\sqrt{x^2 - 25}}{5} - \sec^{-1} \left(\frac{x}{5} \right) \right) + C$$

$$[2] \int e^x \sqrt{1-e^{2x}} dx = \int e^x \sqrt{1-(e^x)^2} dx$$

$$\text{Let } y = e^x$$

$$dy = e^x dx$$

$$dx = \frac{dy}{e^x}$$

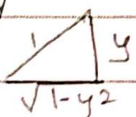
$$= \int y \sqrt{1-y^2} \frac{dy}{y}$$

$$= \int \sqrt{1-y^2} dy$$

$$y = \sin \theta$$

$$dy = \cos \theta d\theta$$

$$\theta = \sin^{-1} y$$



$$= \int \sqrt{1-\sin^2 \theta} \cdot \cos \theta d\theta$$

$$= \int |\cos \theta| \cos \theta d\theta$$

$$= \int \cos^2 \theta d\theta$$

$$= \int \frac{1}{2} (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C$$

$$= \frac{1}{2} \left(\theta + \sin \theta \cos \theta \right) + C$$

$$= \frac{1}{2} \left(\sin^{-1}(y) + \sin(\sin^{-1} y) \cos(\sin^{-1} y) \right) + C$$

$$= \frac{1}{2} \left(\sin^{-1} y + y \sqrt{1-y^2} \right) + C$$

$$= \frac{1}{2} \left(\sin^{-1}(e^x) + 2e^x \sqrt{1-e^{2x}} \right) + C$$

$$\int \frac{\cos t}{\sqrt{2 - \sin^2 t}} dt = \int \frac{\cos t}{2 - y^2} \cdot \frac{dy}{\cos t}$$

$$\text{let } y = \sin t$$

$$dy = \cos t dt$$

$$dt = \frac{dy}{\cos t}$$

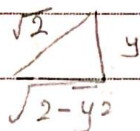
$$= \int \frac{1}{\sqrt{2 - y^2}} dy$$

$$\text{let } y = \sqrt{2} \sin \theta$$

$$dy = \sqrt{2} \cos \theta d\theta$$

$$\frac{y}{\sqrt{2}} = \sin \theta$$

$$\theta = \sin^{-1}\left(\frac{y}{\sqrt{2}}\right)$$



$$= \int \frac{1}{\sqrt{2 - (\sqrt{2} \sin \theta)^2}} \cdot \sqrt{2} \cos \theta d\theta$$

$$= \int \frac{1}{\sqrt{2} \sqrt{1 - \sin^2 \theta}} \cdot \sqrt{2} \cos \theta d\theta$$

$$= \int \frac{\cos \theta}{|\cos \theta|} d\theta$$

$$= \int 1 d\theta$$

$$= \theta + C$$

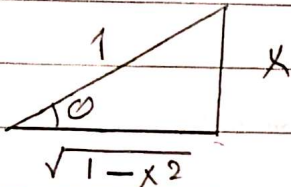
$$= \sin^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$$

$$= \sin^{-1}\left(\frac{\sin t}{\sqrt{2}}\right) + C$$

Example 8-

$$\text{Find } \int \cos(\sin^{-1}x) dx = \int \sqrt{1-x^2} dx$$

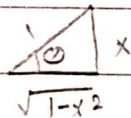
$$\sin(\sin^{-1}x) = x$$



$$\text{let } x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$\theta = \sin^{-1}(x)$$



$$= \int \sqrt{1-\sin^2 \theta} \cdot \cos \theta d\theta$$

$$= \int \sqrt{\cos^2 \theta} \cdot \cos \theta d\theta$$

$$= \int |\cos \theta| \cdot \cos \theta d\theta$$

$$= \int \frac{1}{2} (1 + \cos(2\theta)) d\theta$$

$$= \frac{1}{2} \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C$$

$$= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C$$

$$= \frac{1}{2} (\sin^{-1}(x) + \sin(\sin^{-1}(x)) \cos(\sin^{-1}(x))) + C$$

$$= \frac{1}{2} (\sin^{-1}(x) + x \sqrt{1-x^2}) + C$$

1 + x^2 does not work
 (x^2 + 2x + 2) does not work
 (x^2 + 2x + 1) + 1 = (x+1)^2 + 1

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{1}{\sqrt{-(x^2-2x)}} dx$$

$$\left(\frac{1}{2} x^{\frac{1}{2} \times 2}\right)^2 = \int \frac{1}{\sqrt{-(x^2-2x+1-1)}} dx$$

$$\left(\frac{1}{2} \times 2\right)^2 = 1$$

$$= \int \frac{1}{\sqrt{-(x-1)(x-1)-1}} dx$$

Let $y = x-1$

$dy = dx$

$$= \int \frac{1}{\sqrt{1-(x-1)^2}} dx$$

$$= \int \frac{1}{\sqrt{1-y^2}} dy$$

$$= \sin^{-1}(y) + C$$

$$= \sin^{-1}(x-1) + C$$

Q3) The most proper trigonometric substitution to solve

$$\int \frac{x}{x^2+2x+2} dx \text{ is } \int \frac{x}{(x^2+2x+1)+1} dx$$

$$\left(\frac{1}{2} x^{\frac{1}{2} \times 2}\right)^2 = \left(\frac{1}{2} \times 2\right)^2 = 1$$

a) $x = \tan \theta$

b) $x = 1 - \tan \theta$

c) $x = \tan \theta - 1$ ←

d) $x = 2 + \tan \theta$

$$\int \frac{x}{(x^2+2x+1)+1} dx$$

$$\int \frac{x}{(x+1)^2+1} dx$$

$$\text{Let } y = x+1$$

$$dy = dx$$

$$y-1 = x$$

$$y = \tan \theta$$

$$x+1 = \tan \theta$$

$$x = \tan \theta - 1$$

$$\int \frac{x}{(x+1)^2 + 1} dx$$

$$\int \frac{y-1}{y^2+1} dy$$

Integration of Rational Function by partial fractions lecture 11

* Integration of rational function by partial fractions

$$\int \frac{f}{g} dx \text{ when } \frac{f}{g} \text{ rational function}$$

$$\text{degree } f < \text{degree } g$$

$$\text{degree } f \geq \text{degree } g$$

$$\text{I] degree } f < \text{degree } g \text{ :-}$$

Example :-

$$\text{I] } \int \frac{1}{x^2+1} dx = \tan^{-1} x + C$$

$$\begin{aligned} \text{II] } \int \frac{3}{x+1} dx &= 3 \int \frac{1}{x+1} dx \\ &= 3 \ln|x+1| + C \end{aligned}$$

* partial fractions :-

$$\text{I] } \frac{f}{g} = \frac{f}{(ax+b)(cx+d)(ex+f)\dots} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f} + \dots$$

$$\text{II] } \frac{f}{g} = \frac{f}{(ax+b)^2 (cx+d)\dots} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \frac{C}{cx+d} + \dots$$

Integration of Rational Function by partial fractions lecture 11

$$\boxed{3} \quad \frac{f}{(ax^2+bx+c)(a_2x^2+b_2x+c_2)} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{a_2x^2+b_2x+c_2}$$

تحلل ↗

$$\boxed{4} \quad \frac{f}{(ax^2+bx+c)^2(ax^2+bx+c)} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{Ex+F}{ax^2+bx+c}$$

Example 8- Write out the form of the partial fraction decomposition of e-

$$\boxed{1} \quad \frac{7x+1}{x^3-x^2} = \frac{7x+1}{x^2(x+1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\boxed{2} \quad \frac{x^3-3x}{(x^2-2)^2} = \frac{x^3-3x}{(x-\sqrt{2})^2(x+\sqrt{2})^2}$$

$$= \frac{A}{x-\sqrt{2}} + \frac{B}{(x-\sqrt{2})^2} + \frac{C}{x+\sqrt{2}} + \frac{D}{(x+\sqrt{2})^2}$$

Integration of Rational Function by partial fractions lecture 11

$$\text{[3]} \quad \frac{5}{(x^2+5)^3(x-1)^3 x^2}$$

$$= \frac{Ax+B}{x^2+5} + \frac{Cx+D}{(x^2+5)^2} + \frac{Ex+F}{(x^2+5)^3}$$

$$+ \frac{G}{x-1} + \frac{H}{(x-1)^2} + \frac{I}{(x-1)^3} + \frac{J}{x} + \frac{K}{x^2}$$

$$\text{[4]} \quad \frac{3x^2+x+4}{x^4+3x^2+2} = \frac{3x^2+x+4}{(x^2+1)(x^2+2)}$$

$$= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$$\text{[5]} \quad \frac{2x}{(x^2-1)(x^2-1)} = \frac{2x}{(x-1)(x^2+x+1)(x-1)(x+1)}$$

$$= \frac{2x}{(x-1)^2(x^2+x+1)(x+1)}$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{E}{x+1}$$

Example:-

$$\text{ii)} \int \frac{3x+8}{x^3+5x^2+6x} dx$$

$\swarrow$
 $x(x^2+5x+6)$
 $x(x+3)(x+2)$

$$= \frac{A}{x} + \frac{B}{(x+3)} + \frac{C}{x+2}$$

$$3x+8 = A(x+3)(x+2) + B(x)(x+2) + C(x)(x+3)$$

$$x=0 \Rightarrow 8 = (3)(2)A$$

$$6A = 8 \Rightarrow A = \frac{8}{6} = \frac{4}{3}$$

$$x = -2 \Rightarrow -6 + 8 = (-2)(1)C$$

$$2 = -2C$$

$$C = -1$$

$$x = -3 \Rightarrow -9 + 8 = (-3)(-1)B$$

$$-1 = 3B$$

$$B = -1/3$$

$$= \int \frac{\frac{4}{3}}{x} + \frac{-\frac{1}{3}}{x+3} + \frac{-1}{x+2} dx$$

$$\frac{4}{3} \ln|x| - \frac{1}{3} \ln|x+3| - \ln|x+2| + C$$

$$\text{Ex 2} \int_2^4 \frac{dx}{x^2(x^2-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}$$

$\downarrow$
 $x^2(x-1)(x+1)$

$$1 = A(x-1)(x+1)(x) + B(x-1)(x+1) + C(x+1)x^2 + D(x-1)x^2$$

$$x=0 \Rightarrow 1 = (-1)(1)B$$

$$B = -1$$

$$x=1 \Rightarrow 1 = (2)C$$

$$C = \frac{1}{2}$$

$$x=-1 \Rightarrow 1 = -2D$$

$$D = -\frac{1}{2}$$

$$x=2 \Rightarrow (1)(3)(2)A + (1)(3)B^{\frac{-3}{2}} + 3(4)C^{\frac{1}{2}} + 4D^{\frac{-1}{2}} = 1$$

$$6A - 3 + 6 - 2 = 1$$

$$6A + 1 = 1$$

$$6A = 0$$

$$A = 0$$

$$= \int_2^4 \frac{-1}{x^2} + \frac{1/2}{x-1} + \frac{-1/2}{x+1} dx$$

$$= \left[\frac{+2}{x} + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| \right]_2^4$$

$$= \left(\frac{1}{2} + \frac{1}{2} \ln|3| - \frac{1}{2} \ln|5| \right)$$

$$- \left(1 - \frac{1}{2} \ln|3| \right)$$

$$= \frac{1}{2} + \frac{1}{2} \ln(3) - \frac{1}{2} \ln(5) - 1 + \frac{1}{2} \ln(3)$$

$$= -\frac{1}{2} + \ln(3) - \frac{1}{2} \ln(5)$$

$$= -\frac{1}{2} + \ln(3) - \ln\sqrt{5}$$

$$= -\frac{1}{2} + \ln\left(\frac{3}{\sqrt{5}}\right)$$

$$\pi_1 \int \frac{x^3 + 3x^2 + x + 9}{(x^2 + 1)(x^2 + 3)} dx$$

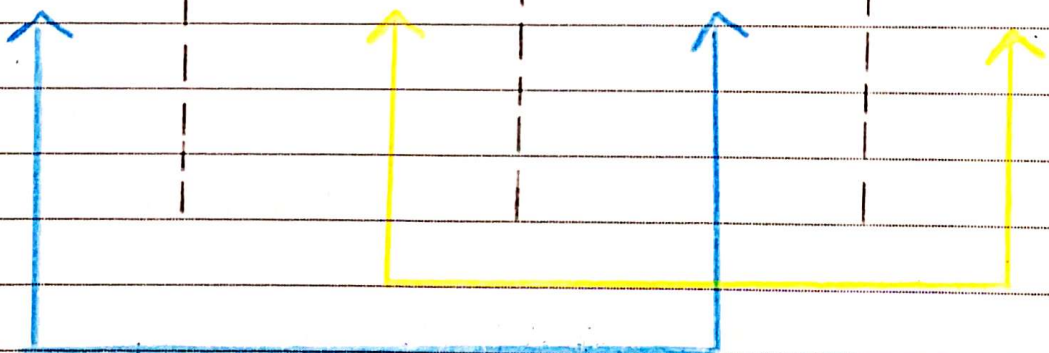
$$= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$$

$$\Rightarrow (Ax+B)(x^2+3) + (Cx+D)(x^2+1) = x^3 + 3x^2 + x + 9$$

$$Ax^3 + 3Ax + Bx^2 + 3B + Cx^3 + Cx + Dx^2 + D = m$$

$$(A+c)x^3 + (B+D)x^2 + (3A+c)x + 3B+D = w$$

$(A+c) = 1$	$B+D=3$	$3A+c=1$	$3B+D=9$
$A=1-c$	$B+9-3B=3$	$3-3c+c=1$	$D=9-3B$
$A=0$	$-2B=-6$	$-2c=-2$	$D=9-9$
	$B=3$	$C=1$	$D=0$



$$\int \frac{3}{x^2+1} dx + \int \frac{x}{x^2+3} dx$$

$$= 3 \int \frac{1}{1+x^2} dx + \frac{1}{2} \int \frac{2x}{x^2+3} dx$$

$$= 3 \tan^{-1}(x) + \frac{1}{2} \ln|x^2+3| + C$$

$$12) \int \frac{e^x}{(e^x-2)(e^x+1)} dx$$

$$\begin{aligned} \text{let } y &= e^x \\ dy &= e^x dx \\ dx &= \frac{dy}{e^x} \end{aligned}$$

$$= \int \frac{1}{(y-2)(y^2+1)} dy$$

$$= \frac{A}{y-2} + \frac{By+C}{y^2+1}$$

$$A(y^2+1) + (By+C)(y-2) = 1$$

$$Ay^2 + A + By^2 - 2By + Cy - 2C = 1$$

$$(A+B)y^2 + (-2B+C)y + (A-2C) = 1$$

$$A+B=0$$

$$-2B+C=0$$

$$A-2C=1$$

$$A=-B$$

$$2+4C+C=0$$

$$A=1+2C$$

$$B=-1-2C$$

$$5C=-2$$

$$A=1+2\left(-\frac{2}{5}\right)$$

$$B=-\frac{5}{5}+\frac{4}{5}$$

$$C=-\frac{2}{5}$$

$$A=\frac{5}{5}-\frac{4}{5}$$

$$B=-\frac{1}{5}$$

$$A=\frac{1}{5}$$

$$\int \frac{\frac{1}{5}}{y-2} dy + \int \frac{-\frac{1}{5}y - \frac{2}{5}}{y^2+1} dy$$

$$= \frac{1}{5} \int \frac{1}{y-2} dy - \frac{1}{5} \int \frac{y}{y^2+1} dy - \frac{2}{5} \int \frac{1}{y^2+1} dy$$

$$= \frac{1}{5} \ln|y-2| - \frac{1}{10} \ln|y^2+1| - \frac{2}{5} \tan^{-1}(y) + C$$

$$= \frac{1}{5} \ln|e^x-2| - \frac{1}{10} \ln|e^{2x}+1| - \frac{2}{5} \tan^{-1}(e^x) + C$$

12) $\int \frac{f}{g}$, degree $f >$ degree g

$$\begin{array}{r} g \overline{) f} \\ \hline \end{array}$$

$$\frac{f}{g} = q + \frac{r}{g}$$

Example 8-

$$\text{11) } \int \frac{x^2 - 7}{x+3} dx$$

$$\begin{array}{r} x-3 \\ x+3 \overline{) x^2 - 7} \\ \underline{-x^2 + 3x} \\ -3x - 7 \\ \underline{+3x + 9} \\ 2 \end{array}$$

$$\int \frac{x^2 - 7}{x+3} dx = \int x - 3 + \frac{2}{x+3} dx$$

$$\frac{1}{2}x^2 - 3x + 2 \ln|x+3| + C$$

part 4

lec type 1

$$[2] \int \frac{x^4 + 7x^3 + 12x^2 + 11x}{x^2 + 2x + 2} dx$$

$$\begin{array}{r} x^2 + 5x - 10 \\ x^2 + 2x + 2 \overline{) x^4 + 7x^3 + 12x^2 + 11x} \\ \underline{-x^4 - 2x^3 - 2x^2} \\ 5x^3 + 10x^2 + 11x \\ \underline{-5x^3 - 10x^2 - 10x} \\ x \end{array}$$

3.1.1

$$= \int x^2 + 5x + \frac{x}{x^2 + 2x + 2} dx$$

$$= \int x^2 + 5x dx + \int \frac{x}{x^2 + 2x + 2} dx$$

$$= \frac{1}{3}x^3 + \frac{5}{2}x^2 + \frac{1}{2} \ln |(x+1)^2 + 1| - \tan^{-1}(x) + C$$

$$1 = (x+1)^2 + 1$$

$$\int \frac{x}{(x^2 + 2x + 1) - 1 + 2} dx$$

$$\int \frac{x}{(x+1)^2 + 1} dx$$

$$y = x + 1$$

$$x = y - 1$$

$$dx = dy$$

$$\int \frac{y-1}{y^2+1} dy$$

$$\int \frac{y}{y^2+1} dy - \int \frac{1}{y^2+1} dy = \frac{1}{2} \ln |y^2+1| - \tan^{-1}(y)$$

Example e-

$$\pi \int \frac{7 \cos \theta}{\sin^2 \theta + 5 \sin \theta - 6} d\theta = \int \frac{7 \cos \theta}{y^2 + 5y - 6} \frac{dy}{\cos \theta}$$

$$\text{let } y = \sin \theta$$

$$dy = \cos \theta d\theta$$

$$\frac{dy}{\cos \theta} = d\theta$$

$$= 7 \int \frac{1}{(y+6)(y-1)} dy$$

$$\frac{A}{y+6} + \frac{B}{y-1}$$

$$A(y-1) + B(y+6) = 1$$

$$y=1 \Rightarrow 7B=1$$

$$B = 1/7$$

$$y=-6 \Rightarrow -7A=1$$

$$A = -1/7$$

$$= \int \frac{-1/7}{y+6} + \frac{1/7}{y-1} dy$$

$$-\frac{1}{7} \ln |y+6| + \frac{1}{7} \ln |y-1| + C$$

$$-\frac{1}{7} \ln |\sin \theta + 6| + \frac{1}{7} \ln |\sin \theta - 1| + C$$

$$Q2) \int \frac{5 + 2 \ln x}{x(1 + \ln x)^2} dx$$

$$\text{let } y = \ln x$$

$$dy = \frac{1}{x} dx$$

$$\int \frac{5 + 2y}{x(1 + y)^2} x dy$$

$$dx = x dy$$

$$= \int \frac{5 + 2y}{y^2 + 2y + 1} dy$$

$$\Rightarrow \frac{A}{y+1} + \frac{B}{(y+1)^2}$$

$$A(y+1) + B = 5 + 2y$$

$$y = -1 \Rightarrow B = 5 - 2$$

$$B = 3$$

$$y = 0 \Rightarrow A + 3 = 5$$

$$A = 2$$

$$\int \frac{2}{y+1} + \frac{3}{(y+1)^2} dy$$

$$= 2 \ln|y+1| + \frac{-3}{y+1} + C$$

$$= 2 \ln|\ln x + 1| + \frac{-3}{\ln x + 1} + C$$

$$13) \int (\sin^{-1} x)^2 dx$$

$$u = (\sin^{-1} x)^2 \quad dv = dx$$

$$du = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} dx \quad v = x$$

-P

$$= x (\sin^{-1} x)^2 - 2 \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

$$u = \sin^{-1} x \quad dv = \frac{x}{\sqrt{1-x^2}}$$

$$du = \frac{1}{\sqrt{1-x^2}} \quad v = -\sqrt{1-x^2}$$

-P

$$= x (\sin^{-1} x)^2 + 2 \sin^{-1} x \sqrt{1-x^2} - 2x + C$$

$$= \sin^{-1} x \sqrt{1-x^2} + \int 1 dx$$

$$= \sin^{-1} x \sqrt{1-x^2} + x + C$$

$$14) \int \frac{1}{\sqrt{x+1}} dx$$

$$\int \frac{1}{y} \cdot 4y \sqrt{x} dy$$

$$4 \int (y^2 - 1) dy$$

$$4 \left(\frac{y^3}{3} - y \right) + C$$

$$4 \left(\frac{1}{3} (\sqrt{x+1})^3 - (\sqrt{x+1}) \right) + C$$

let $y = \sqrt{x+1}$

$$y^2 = x+1$$

$$2y dy = \frac{1}{2\sqrt{x}} dx$$

$$dx = 4y \sqrt{x} dy$$

$$\sqrt{x} = y^2 - 1$$

Example:-

$$Q1 \int \frac{x e^x}{(1+x)^2} dx$$

$$= \frac{-x e^x}{x+1} + \int \frac{e^x + x e^x}{x+1} dx$$

$$u = x e^x$$

$$dV = (1+x)^{-2}$$

$$du = e^x + x e^x$$

$$V = -(1+x)^{-1}$$

-f

$$= \frac{-x e^x}{x+1} + \int \frac{e^x (x+1)}{x+1} dx$$

$$= \frac{-x e^x}{x+1} + \int e^x dx$$

$$= -\frac{x e^x}{x+1} + e^x + C$$

$$Q2 \int \ln(x + \sqrt{x^2 - 1}) dx$$

$$u = \ln(x + \sqrt{x^2 - 1})$$

$$dV = dx$$

$$du = \frac{1 + \frac{2x}{\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}}$$

$$V = x$$

-f

$$= x \ln(x + \sqrt{x^2 - 1}) - \int \frac{x (\sqrt{x^2 - 1} + 2x)}{\sqrt{x^2 - 1} (x + \sqrt{x^2 - 1})} dx$$

last

$$13) \int e^{\sqrt{x}} dx$$

$$= \int e^y 2y dy$$

$$y = \sqrt{x}$$

$$y^2 = x$$

$$2y dy = dx$$

u	dv
2y	e ^y
2	e ^y
0	e ^y

+P

$$\int e^y 2y dy = 2y e^y - 2e^y + \int 0 dy$$

$$= 2y e^y - 2e^y + C$$

$$= 2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

$$14) \int \frac{\sqrt{x}}{1+\sqrt[3]{x}} dx$$

Let's $y = \sqrt{x}$
 $x = y^2$
 $dx = 2y dy$

$$= \int \frac{\sqrt{y^4}}{1+\sqrt[3]{y^2}} dy 2y$$

$$= 2 \int \frac{y^2}{1+y^{2/3}} y dy$$

$$= 2 \int \frac{y^8}{1+y^2} dy$$

$$\begin{array}{r} y^6 - y^4 \\ y^2 + 1 \overline{) y^8} \\ \underline{-y^8 + y^6} \\ y^6 - y^4 \\ \underline{+y^6 + y^4} \\ 4y^4 \end{array}$$

نفرض $\sqrt[5]{x}, \sqrt[3]{x} \Rightarrow x = y^6$

نفرض $\sqrt[4]{x}, \sqrt{x} \Rightarrow x = y^8$

نفرض $\sqrt{x}, \sqrt[3]{x} \Rightarrow x = y^6$

$$= \int \frac{y^6 - y^4 + y^2 - 1}{y^2 + 1} dy$$

$$\begin{array}{r} y^6 - y^4 + y^2 - 1 \\ y^2 + 1 \overline{) y^8 - y^6 + y^4 - y^2 - 1} \\ \underline{-y^8 - y^6} \\ -y^6 + y^4 - y^2 - 1 \\ \underline{+y^6 + y^4} \\ y^4 - y^2 - 1 \\ \underline{-y^4 + y^2} \\ -y^2 - 1 \\ \underline{+y^2 + 1} \\ 0 \end{array}$$

$$= \frac{1}{7} y^7 - \frac{1}{5} y^5 + \frac{1}{3} y^3 - y + \tan^{-1} y + C$$

$$= \frac{1}{7} (\sqrt[6]{x})^7 - \frac{1}{5} (\sqrt[6]{x})^5 + \frac{1}{3} (\sqrt[6]{x})^3 - (\sqrt[6]{x}) + \tan^{-1}(\sqrt[6]{x}) + C$$

$$15) \int \frac{1}{1+e^x} dx$$

$$\int \frac{1}{1+y} \frac{dy}{y}$$

$$y = e^x$$

$$dy = e^x dx$$

$$dx = \frac{dy}{e^x} = \frac{dy}{y}$$

$$\int \frac{1}{y(y+1)} dy$$

$$\Rightarrow \frac{A}{y} + \frac{B}{y+1}$$

$$A(y+1) + B(y) = 1$$

$$= \int \frac{1}{y} + \frac{-1}{y+1} dy$$

$$y=0 \Rightarrow \boxed{A=1}$$

$$y=-1 \Rightarrow -B=1$$

$$\boxed{B=-1}$$

$$= \ln|y| - \ln|y+1| + C$$

$$= \ln|e^x| - \ln|e^x+1| + C$$

$$Q1 \int \frac{\sqrt{x+4}}{x} dx$$

$$= \int \frac{y}{y^2-4} 2y dy$$

$$= \int \frac{2y^2}{y^2-4} dy$$

$$\text{let } y = \sqrt{x+4}$$

$$y^2 = x+4$$

$$x = y^2 - 4$$

$$dx = 2y dy$$

$$= \int 2 + \frac{8}{y^2-4} dy \quad \left| \frac{y^2-4}{2y^2} \right| \frac{2y^2}{-2y^2+8} +$$

$$= 2y + 8 \int \frac{1}{(y-2)(y+2)} dy$$

$$= 2y + 8 \int \frac{1/4}{y-2} - \frac{1/4}{y+2} dy$$

$$\frac{A}{y-2} + \frac{B}{y+2}$$

$$= 2y + 8 \left(\frac{1}{4} \ln|y-2| - \frac{1}{4} \ln|y+2| \right) \quad A(y+2) + B(y-2) = 1$$

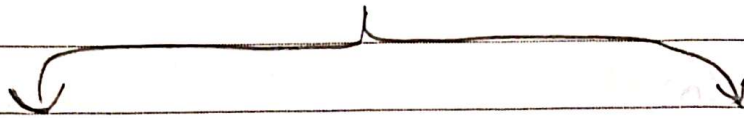
$$y=2 \Rightarrow 4A=1 \Rightarrow A=1/4$$

$$y=-2 \Rightarrow -4B=1 \Rightarrow B=-1/4$$

$$= 2y + 2 \ln|y-2| - 2 \ln|y+2| + C$$

$$= 2\sqrt{x+4} + 2 \ln|\sqrt{x+4}-2| - 2 \ln|\sqrt{x+4}+2| + C$$

Improper Integrals



Type 1: Infinite Intervals

$$\int_a^{\infty} f(x) dx, \int_{-\infty}^b f(x) dx, \int_{-\infty}^{\infty} f(x) dx$$

Type 2: Discontinuous Integrands

(f has vertical asymptote)

$$\int_a^b f(x) dx$$

Example 8-

$$\text{I}_1 \int_1^{\infty} e^{-x} dx$$

improper integrals (∞)

$$\text{I}_2 \int_2^3 \frac{1}{x^2} dx$$

not improper

$$\text{I}_3 \int_0^{\pi/2} \tan x dx$$

improper (discont at $\pi/2$)

$$\text{I}_4 \int_{-\infty}^5 \frac{1}{x-2} dx$$

improper ($-\infty$, discont at $x=2$)

$$\text{I}_5 \int_0^1 \ln x dx$$

improper (discont at $x=0$)

$$\text{I}_6 \int_1^5 \frac{dx}{x+2}$$

not improper integral

F1) Definition of an improper Integral of Type 1

(a) If $\int_a^t f(x) dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

provided this limit exists (as a finite number)

b) If $\int_t^b f(x) dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

provided this limit exists (as a finite number)

The improper integrals $\int_a^{\infty} f(x) dx$ and $\int_{-\infty}^b f(x) dx$ are called **convergent**

if the corresponding limits exist and **divergent** if the limit does not exist.

c) If both $\int_a^{\infty} f(x) dx$ and $\int_{-\infty}^a f(x) dx$ are convergent

then we define

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx$$

* In part c any real number a can be used

Example:-

$$\pi) \int_{-1}^{\infty} \frac{x}{1+x^2} dx$$

$$= \lim_{t \rightarrow \infty} \int_{-1}^t \frac{x}{1+x^2} dx$$

$$= \lim_{t \rightarrow \infty} \left[\frac{1}{2} \ln |1+x^2| \right]_{-1}^t$$

$$= \lim_{t \rightarrow \infty} \frac{1}{2} \ln(1+t^2) - \frac{1}{2} \ln(2)$$

$$= \frac{1}{2} \lim_{t \rightarrow \infty} \ln \frac{1}{2} (1+t^2) - \frac{1}{2} \ln 2$$

$$= \frac{1}{2} \ln \left(\lim_{t \rightarrow \infty} \frac{1}{2} (1+t^2) \right) - \frac{1}{2} \ln 2$$

$$= \infty$$

~~∞~~ divergent

$$\text{Ex } \int_{-\infty}^0 e^{4x} dx$$

$$= \lim_{t \rightarrow -\infty} \int_t^0 e^{4x} dx$$

$$= \frac{1}{4} \lim_{t \rightarrow -\infty} 1 - e^{4t}$$

$$= \frac{1}{4} \lim_{t \rightarrow -\infty} 1 - e^{4t}$$

$$= \frac{1}{4} (1 - 0)$$

$$= \frac{1}{4}$$

$\therefore$ Converge to $\frac{1}{4}$

Example 9:

$$\text{ii) } \int_0^{\infty} (1-x) e^{-x} dx$$

$$= \lim_{t \rightarrow \infty} \int_0^t (1-x) e^{-x} dx$$

$$= \lim_{t \rightarrow \infty} \left[-(1-x) e^{-x} + e^{-x} \right]_0^t$$

Integration by parts diagram:

u	dv
$1-x$	e^{-x}
-1	$-e^{-x}$
0	e^{-x}

Result: $+$

$$= \lim_{t \rightarrow \infty} \left((-e^{-t}(1-t) + e^{-t}) - (-1 + 1) \right)$$

Zero

$$= \lim_{t \rightarrow \infty} \left((t-1) e^{-t} + e^{-t} \right)$$

$$\lim_{t \rightarrow \infty} e^{-t} ((t-1) + 1)$$

$$\lim_{t \rightarrow \infty} e^{-t} t \quad (\infty \cdot 0)$$

$$\lim_{t \rightarrow \infty} \frac{t}{e^t} = \lim_{t \rightarrow \infty} \frac{1}{e^t} = 0$$

Converge to Zero

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

نعمل
كل تكامل
على حدة

$$= \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx$$

نفرق
أي عو

$$= \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{1+x^2} dx + \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1+x^2} dx$$

$$= \lim_{t \rightarrow -\infty} \left[\tan^{-1} x \right]_t^0 + \lim_{t \rightarrow \infty} \left[\tan^{-1} x \right]_0^t$$

$$= \lim_{t \rightarrow -\infty} (\tan^{-1}(0) - \tan^{-1}(t)) + \lim_{t \rightarrow \infty} (\tan^{-1}(t) - \tan^{-1}(0))$$

$$= \lim_{t \rightarrow -\infty} -\tan^{-1}(t) + \lim_{t \rightarrow \infty} \tan^{-1}(t)$$

$$= -\frac{\pi}{2} + \frac{\pi}{2}$$

$$= \frac{2\pi}{2}$$

$$= \pi$$

~~✗~~ Converge to π

Improper Integrals part 2

lecture 18

Theorem 8 $\int_1^{\infty} \frac{1}{x^p} dx$ $\begin{cases} \text{converge to } \frac{1}{p-1} & \text{if } p > 1 \\ \text{Diverge} & \text{if } p \leq 1 \end{cases}$

Example- $\int_1^{\infty} x^{-3} dx = \int_1^{\infty} \frac{1}{x^3} dx$ $\begin{matrix} p=3 \\ p > 1 \end{matrix}$

* converge to $\frac{1}{3-1} = \frac{1}{2}$

Example- Find the value of p for which the integral converge and evaluate the integral for those value of p .

1) $\int_1^{\infty} \frac{1}{x^{p-2}} dx$ $\begin{cases} \text{converge if } p-2 > 1 \\ \phantom{\text{converge if }} p > 3 \\ \phantom{\text{converge if }} p \in (3, \infty) \\ \text{diverge if } p \leq 3 \\ \phantom{\text{diverge if }} p \in (-\infty, 3] \end{cases}$

$\int_1^{\infty} \frac{1}{x^{p-2}} dx$ converge to $\frac{1}{p-2-1} = \frac{1}{p-3}$

$$[2] \int_e^{\infty} \frac{1}{x (\ln x)^{2p}} dx$$

let $y = \ln x$

$$dy = \frac{1}{x} dx$$

$$\int_1^{\infty} \frac{1}{x y^{2p}} x dy$$

$$x dy = dx$$

$$\int_1^{\infty} \frac{1}{y^{2p}} dy$$

$$x = e \Rightarrow y = \ln e = 1$$

$$x = \infty \Rightarrow y = \infty$$

diverge if $2p < 1$ $\leftarrow$ converge to $\frac{1}{2p-1}$ if $2p > 1$
 $p < \frac{1}{2}$ $p > \frac{1}{2}$

3) Definition of an Improper Integrals of Type 2

(a) If f is continuous on $[a, b]$ and is discontinuous at b , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

If this limit exists (as a finite number)

(b) If f is continuous on $(a, b]$ and is discontinuous at a , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

If this limit exists (as a finite number)

The improper integral $\int_a^b f(x) dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If f has a discontinuity at c , where $a < c < b$, and both $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ are convergent, then we define

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Example

$$ii) \int_0^{\pi/2} \cot x \, dx = \int_0^{\pi/2} \frac{\cos x}{\sin x} \, dx$$

$$= \lim_{t \rightarrow 0^+} \int_t^{\pi/2} \frac{\cos x}{\sin x} \, dx$$

$$= \lim_{t \rightarrow 0^+} \left[\ln |\sin x| \right]_t^{\pi/2}$$

$$= \lim_{t \rightarrow 0^+} \left(\ln(\sin \pi/2) - \ln(\sin t) \right)$$

$$= \lim_{t \rightarrow 0^+} 0 - \ln(\sin t)$$

$$= -\ln \lim_{t \rightarrow 0^+} \sin t$$

$$= -(-\infty) = +\infty$$

$$ii) \int_0^{\pi/2} \cot x \, dx \text{ divergent}$$

Improper Integral part 3

lecture 19

$$\text{Ex 1) } \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

$$= \lim_{t \rightarrow 1^-} \int_0^t \frac{1}{\sqrt{1-x^2}} dx$$

$$= \lim_{t \rightarrow 1^-} \left[\sin^{-1}(x) \right]_0^t$$

$$= \lim_{t \rightarrow 1^-} \sin^{-1}(t) - \cancel{\sin^{-1}(0)}$$

$$= \lim_{t \rightarrow 1^-} \sin^{-1}(t)$$

$$= \pi/2$$

~~converge~~ to $\pi/2$

$$\text{Ex 2) } \int_{-2}^2 \frac{1}{x^2} dx = \int_{-2}^0 x^{-2} dx + \int_0^2 x^{-2} dx$$

$$= \lim_{t \rightarrow 0^-} \int_{-2}^t x^{-2} dx + \sim$$

$$= \lim_{t \rightarrow 0^-} \left[-x^{-1} \right]_{-2}^t + \sim$$

$$= \lim_{t \rightarrow 0^-} -t^{-1} + \frac{1}{2} + \sim$$

$$= -(-\infty) + \sim$$

does not exist

~~converge~~ divergent

$$\text{Ex } \int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(x+1)}$$

$$= \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{\sqrt{x}(x+1)} dx + \lim_{t \rightarrow \infty} \int_1^t \frac{1}{\sqrt{x}(x+1)}$$

let $y = \sqrt{x}$

$y^2 = x$

$2y dy = dx$

$$= \lim_{t \rightarrow 0^+} \int \frac{1}{y(y^2+1)} 2y dy + \lim_{t \rightarrow \infty} \int \frac{2}{y^2+1}$$

$$= \lim_{t \rightarrow 0^+} \int \frac{2}{y^2+1} dy + \lim_{t \rightarrow \infty} \left[2 \tan^{-1}(\sqrt{x}) \right]_1^t$$

$$= \lim_{t \rightarrow 0^+} \left[2 \tan^{-1}(\sqrt{x}) \right]_t^1 + \lim_{t \rightarrow \infty} 2 \left(\tan^{-1}(\sqrt{t}) - \tan^{-1}(1) \right)$$

$$= \lim_{t \rightarrow 0^+} 2 \left(\tan^{-1}(1) - \tan^{-1}(\sqrt{t}) \right) + 2 \frac{\pi}{2} - 2 \frac{\pi}{4}$$

$$= 2 \frac{\pi}{4} + \pi - \frac{\pi}{2}$$

$$= \frac{\pi}{2} + \pi - \frac{\pi}{2}$$

$$= \pi$$

converge to π

Example 4 Find the set of values of p for which the integral is improper $\int_0^9 \frac{1}{\sqrt{x-p}} dx$?

$$\sqrt{x} - p = 0$$

$$p = \sqrt{x}$$

$$0 \leq x \leq 9$$

$$0 \leq \sqrt{x} \leq 3$$

$$0 \leq p \leq 3$$

$$p \in [0, 3]$$